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Finiteness of flux vacua from geometric transitions

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Abstract: We argue for finiteness of flux vacua around type IIB CY singularities by computing their gauge theory duals. This leads us to propose a geometric transition where the compact 3-cycles support both RR and NS flux, while the open string side contains 5-brane bound states. By a suitable combination of S duality and symplectic transformations, both sides are shown to have the same IR physics. The finiteness then follows from a holomorphic change of couplings in the gauge side. As a nontrivial test, we compute the number of vacua on both sides for the conifold and the Argyres-Douglas point, and we find perfect agreement.

Keywords: Gauge-gravity correspondence, Flux compactifications, Superstring Vacua.

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Contents

1. Introduction

The study of string vacua in flux compactifications of type IIB has attracted much attention, in part as a setting where many properties of the landscape of vacua are under control (see [1] for a recent review). One of the first issues to be addressed here is whether the number of realistic string vacua is finite [2]. Recently it has been shown in [3] that the Ashok-Douglas index of supersymmetric vacua is finite. This is a crucial step towards proving the finiteness of flux vacua.

The aim of the present work is to understand the physics underlying the previous result. It was shown in [4] that the index of supersymmetric vacua is finite around smooth points in moduli space; the analysis may be restricted then to singularities of the moduli space where the curvature diverges. The finiteness proof [3] is based on Weil-Petersson geometry and a detailed analysis of degenerations of Hodge structures on the moduli space. However, from the string theory point of view it is not clear which is the physical mechanism responsible for this. Specially, why singularities leading to very different field theories all give a finite number of vacua.

Our approach may be summarized as follows. We construct Calabi-Yau's where singularities are easily embedded and argue for finiteness of vacua around them by computing their dual gauge theories. We establish a precise correspondence between flux and gauge degrees of freedom. This shows that the gauge theories are generalized versions of the ones obtained through the Dijkgraaf-Vafa correspondence [5]; but they still have finitely many vacua. As we shall see, the underlying reason for this is the topological nature of the chiral ring of such theories.

In section 2 we discuss the type IIB noncompact model that can embed ADE singularities and study the nonperturbative superpotential generated by fluxes. Next, in section 3 we derive the formula for counting vacua in the previous setup; this involves nontrivial steps because of the noncompact nature of the model. Then in section 4 we construct the dual gauge theory after the geometric transition, applying S-duality to the Dirac-Born-Infeld (DBI) action. The field theory turns out to be a generalization of the usual $N = 1$ SYM encountered in geometric transitions. The argument for finiteness of vacua is presented in section 5; it is based on the holomorphic dependence of the gauge effective superpotential on nondynamical fields (couplings). Finally, in section 6 we show that the computations from the gravity and gauge side agree for the conifold and Argyres-Douglas singularities. Section 7 contains our conclusions.

2. Fluxes in noncompact Calabi-Yau's

We start by studying moduli stabilization in type IIB theory in a Calabi-Yau threefold. Since we are interested in analyzing a neighborhood of an ADE singularity, it is enough to consider noncompact threefolds of the form

$$P := u^2 + v^2 + F(x, y) = 0\,; \tag{2.1}$$

the nontrivial dynamics comes from the complex curve Σ: $F(x, y) = 0$. (2.1) may be thought as a decoupling limit $M_{Pl} \to \infty$ of an adequate compact variety [6], although this will not be necessary for our purposes.

For concreteness, let us consider the case of a hyperelliptic curve where we can realize singularities of the A-type:

$$F(x, y) = y^2 - W'(x)^2 - f_{n-1}(x) = 0\,. \tag{2.2}$$

$W'(x)$ is a polynomial of degree n, and will play the role of the superpotential in the gauge theory:

$$W'(x) = g_n \prod_{i=1}^{n} (x - a_i)\,. \tag{2.3}$$

$f_{n-1}(x) = \sum_{k=1}^{n} f_k x^{k-1}$ is a deformation of the singular curve $y^2 = W'(x)^2$. Its effect is to split $a_i \to (a_i^-, a_i^+)$. If all the roots of W are different then the singular curve has just ODP (conifold) singularities. We will also encounter more complicated singularities, where three or more roots coincide.

The fact that (2.2) is the same variety that appears in the Dijkgraaf-Vafa duality [5] is not a coincidence; the (generalized) gauge dual will play a major role in proving the finiteness of the number of vacua. Furthermore, such local string models have been considered recently in the context of soft supersymmetry breaking [7].

For our future computations, it is crucial to remark the following. In the four dimensional effective field theory (EFT), the moduli (a_i, f_k) have a very different interpretation. Fluctuations in a_i have infinite energy and hence are non-dynamical; each arbitrary choice of a_i will give a different 4d theory so they can be interpreted as couplings. On the other hand, the f_k's are dynamical and are interpreted as scalar fields in vector multiplets. Their gauge theory meaning will become clear in section 4.

As shown in [8], the periods of the noncompact threefold reduce to periods of the hyperelliptic curve:

$$S_i = \int_{A_i} R(x)dx \;\;,\;\; \frac{\partial \mathcal{F}}{\partial S_i} = 2\pi i \int_{B_i} R(x)dx \tag{2.4}$$

with

$$2R(x) = W'(x) - \sqrt{W'(x)^2 + f_{n-1}(x)}\,. \tag{2.5}$$

The cycle A_i surrounds the cut $[a_i^-, a_i^+]$; B_i is the noncompact cycle dual to A_i, running from $x = a_i$ to infinity. The B-periods need to be regulated; this will be discussed shortly. Therefore all the computations can be done directly on the hyperelliptic curve $y(x)$ of genus $g = n - 1$.

When $x \to \infty$

$$R(x) \to -\frac{f_n}{2g_n x}\,. \tag{2.6}$$

This implies that R (and y) is a differential of the third kind on Σ [9]. For any value $x \in \mathbb{C}$, there are two points on the Riemann surface Σ; let $P, \tilde{P} \in \Sigma$ denote the points corresponding to $x = \infty$. Then $R(x)dx$ is a holomorphic differential only on the punctured surface $\Sigma' = \Sigma - \{P, \tilde{P}\}$.

The details of the homology of Σ and the effect of the punctures were considered in [10] and we follow their conventions. A choice of homology cycles is shown in figure 1; B_j runs through the j-th cut, from $\tilde{P}$ to P. From these noncompact cycles we construct $C_i = B_i - B_n$. Besides, C_P and $C_{\tilde{P}}$ circle the punctures at P and $\tilde{P}$ respectively. The canonical symplectic basis of Σ is (A_i, C_j), $i, j = 1, \ldots, g = n - 1$. In Σ, $A_1 + \ldots + A_n \equiv 0$ so A_n is not independent; however, in Σ', $A_1 + \ldots + A_n = -C_P$. This means that we can take A_n to be an independent cycle and use this to fix the values of the meromorphic differentials at infinity. A symplectic basis for $H_1(\Sigma', \mathbb{Z})$ is hence (A_i, B_j), $i, j = 1, \ldots, n$.

In the holomorphic decomposition $H^1(\Sigma, \mathbb{C}) = H^{1,0}(\Sigma, \mathbb{C}) + H^{0,1}(\Sigma, \mathbb{C})$, there is a unique basis of holomorphic differentials [9] $(\zeta_1, \ldots, \zeta_g)$ such that

$$\int_{A_j} \zeta_k = \delta_{jk} \;\;,\;\; \operatorname{Im}\Pi \geq 0 \tag{2.7}$$

where the period matrix Π is defined to be the symmetric matrix

$$\Pi_{jk} = \int_{C_j} \zeta_k\,.$$

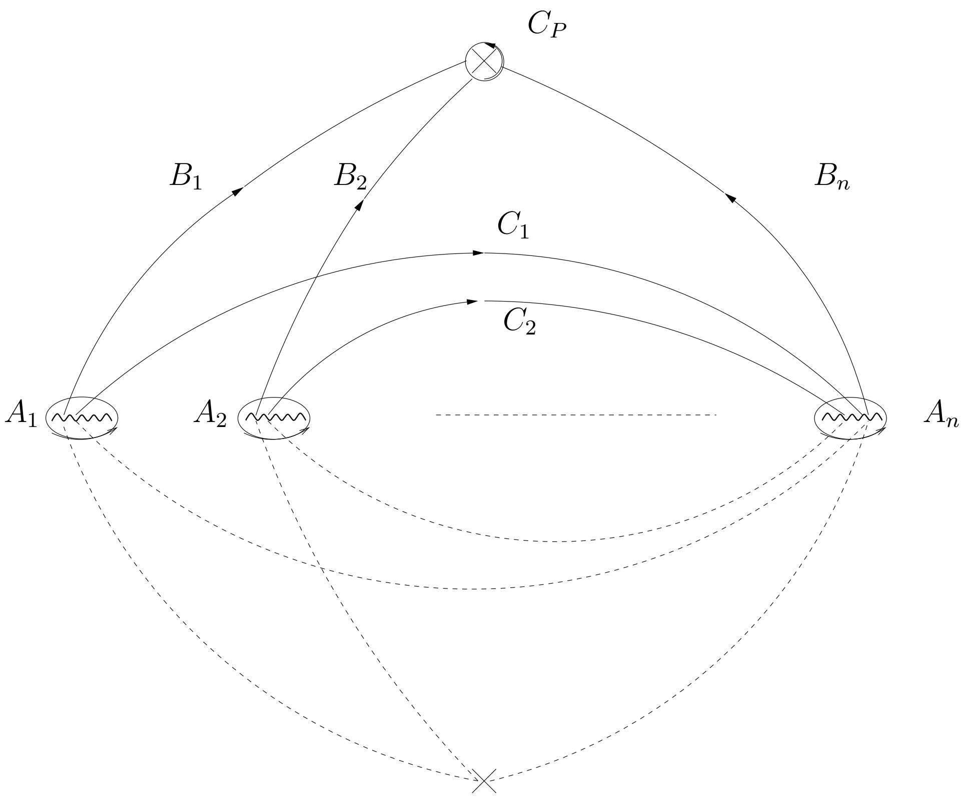

Figure 1: Homology elements of Σ and Σ'.

They can be constructed as linear combinations of the differentials

$$\frac{\partial}{\partial f_k} y\, dx = \frac{x^{k-1}}{2y} dx \quad , \quad k = 1, \ldots, n-1 \,. \tag{2.8}$$

The third kind differential

$$g_n \frac{\partial}{\partial f_n} y\, dx = \frac{g_n x^{n-1}}{2y} dx \tag{2.9}$$

has residues ± 1 at $P, \tilde{P}$ respectively. An adequate linear combination of (2.8) and (2.9) will give the unique third kind differential $\tau_{P,\,\tilde{P}}$ such that

$$\mathrm{ord}_P\, \tau_{P,\,\tilde{P}} = \mathrm{ord}_{\tilde{P}}\, \tau_{P,\,\tilde{P}} = -1 \,,$$

$$\mathrm{res}_P\, \tau_{P,\,\tilde{P}} = 1 \;, \;\; \mathrm{res}_{\tilde{P}}\, \tau_{P,\,\tilde{P}} = -1 \,.$$

Every holomorphic differential on Σ' can be written as a linear combination of $(\zeta_1, \ldots, \zeta_g, \tau_{P\tilde{P}})$. Such differentials are meromorphic differentials on Σ with at most simple poles. A more symmetric description follows from taking A_n (instead of C_P) to be an independent cycle; hence the basis of allowed differentials will be $(\zeta_1, \ldots, \zeta_{n-1}, \zeta_n)$ where ζ_n is a superposition of ζ_i and $\tau_{P,\,\tilde{P}}$ fixed by $\int_{A_j} \zeta_n = \delta_{jn}$, $j = 1, \ldots, n$.

2.1 Superpotential and fluxes

The complex moduli of X are stabilized by turning on 3-form fluxes $G_3 := F_3 - \tau H_3$, which generate the nonperturbative superpotential [11]

$$W_{\text{eff}} = \int_X G_3 \wedge \Omega\,. \tag{2.10}$$

In the noncompact model, the axio-dilaton τ is fixed, corresponding to a coupling of the 4d EFT. Upon integrating over the S^2 fibers given by (u, v), (2.10) reduces to the superpotential on the hyperelliptic curve

$$W_{\text{eff}} = \int_{\Sigma'} T \wedge R\,. \tag{2.11}$$

The fluxes through all the compact cycles are quantized:

$$\int_{A_i} T = N_i^R - \tau N_i^{NS} \quad , \quad \int_{C_i} T = c_i^R - \tau c_i^{NS}\,, \tag{2.12}$$

$N_i^R\,, N_i^{NS}\,, c_i^R\,, c_i^{NS} \ \in \mathbb{Z}$. However, the fluxes through the noncompact cycles can vary continuously and, in fact, we will argue that they have to diverge. We denote

$$-\int_{B_i} T := \beta_i^R - \tau \beta_i^{NS}\,. \tag{2.13}$$

These quantities will play the role of running gauge couplings.

Given that the B-cycles extend to infinity, and both R and T are differentials of the third kind, we need to regulate their B periods. Following [12] we introduce a cut-off at large distances $x = \Lambda_0$, replacing P and $\tilde{P}$ by Λ_0 and $\tilde{\Lambda}_0$. For the noncompact approximation to be consistent, (2.11) has to be finite in the limit $\Lambda_0 \to \infty$. We write B_i^r for the regularized version of B_i, running from $\tilde{\Lambda}_0$ to Λ_0 through the $[a_i^-, a_i^+]$ cut.

The Λ_0 dependence of $\int_{B_i^r} R$ is most easily obtained [8] by doing a monodromy around infinity $\Lambda_0^{3/2} \to \mathrm{e}^{2\pi i}\Lambda_0^{3/2}$.[1] In Σ' this corresponds to $B_i^r \to B_i^r + C_p + C_{\tilde{P}} = B_i^r - 2\sum_{i=1}^n A_i\,$, giving

$$\int_{B_i^r} R = -\frac{1}{2\pi i}\left(\sum_{i=1}^n S_i\right) \log \Lambda_0^3 + \ldots \tag{2.14}$$

where $\ldots$ are single valued contributions. Comparing with (2.6),

$$f_n = -4 g_n \sum_{i=1}^n S_i\,. \tag{2.15}$$

From (2.14), we see that all the periods have the same $\log \Lambda_0^3$ dependence.

It was shown in [8] that the cutoff dependence of T is exactly the one needed to cancel the logarithmic divergence from (2.14) and yield a finite cutoff independent W_{eff}:

$$\beta_i^R - \tau \beta_i^{NS} \;=\; \frac{1}{2\pi i}(N_i^R - \tau N_i^{NS}) \log{(\Lambda_0/\Lambda_i)^3}\,. \tag{2.16}$$

The β_i where defined in (2.13) and Λ_i are a set of finite energy scales. Therefore (2.16) may be interpreted as a *geometric renormalization* of certain bare coupling constants (β_i^R, β_i^{NS}). This is the geometric analog of the RG running of the gauge couplings (see sections 4 and 5).

[1] The exponent is the mass dimension of x: $[x] = 3/2$, which follows from $[S] = 3$.

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3. Counting vacua on curves with punctures

In this section we develop the necessary tools to count supersymmetric flux vacua on the hyperelliptic curve (2.2). We will show that the index formula $\int \det(-R-\omega)$ of [4] is still valid in our case. This is a priori not obvious, the main issues being that the curve is noncompact so many quantities need a regulator and the punctures contribute extra moduli that have to be included. Furthermore, having fluxes β_i that can vary continuously would immediately lead to an infinite number of vacua. And finally, we will have to introduce a tadpole cancellation condition.

Counting supersymmetric flux vacua is equivalent to studying the geometry of the moduli space $\mathcal{M}$ of Σ'. There are different ways of parametrizing it; while from the EFT it is natural to work with the S_i, in the geometrical side it is more convenient to use the coefficients f_k of the deformation $f_{n-1}(x)$. More specifically, we parametrize $\mathcal{M}$ by combinations u_k of the f_k $(k = 1, \ldots, n)$ such that

$$\frac{\partial R}{\partial u_k} = \zeta_k \,,$$

giving directly the basis of holomorphic differentials introduced in (2.7) plus ζ_n. This is an efficient and symmetric way of taking into account the modulus from the puncture at P and will simplify our formulas.

Σ becomes singular when two branch points coincide; this leads us to define the discriminant

$$\Delta(u) := \prod_{a<b}(e_a - e_b)^2 \tag{3.1}$$

where $e_a := a_i^{\pm}$. We denote the zero locus by Σ_Δ; the moduli space is therefore

$$\mathcal{M} = \{(u_k) \in \mathbb{C}^n\} \setminus \Sigma_\Delta \,. \tag{3.2}$$

Σ_Δ is codimension one in $\mathcal{M}$ and corresponds to conifold-like singularities: around two coinciding roots we can always perform a holomorphic change of variables to rewrite the curve as

$$u^2 + v^2 + y^2 - x^2 = 0 \,.$$

Higher order Argyres-Douglas singularities [13] occur when three or more roots coincide, and will be discussed in sections 5 and 6.

The moduli space is a special Kahler manifold, with metric

$$G_{i\bar{l}} = -i \int_{\Sigma'} \zeta_i \wedge \bar{\zeta}_{\bar{l}} \tag{3.3}$$

which can be derived from the Kahler potential

$$K(u, \bar{u}) = -i \int R \wedge \bar{R} \,. \tag{3.4}$$

The covariant derivative is

$$\nabla_i V^j = \partial_i V^j + \Gamma^j{}_{ik} V^k \quad , \quad \Gamma^j{}_{ik} = G^{j\bar{l}} \partial_i G_{k\bar{l}} \tag{3.5}$$

($\partial_i := \partial/\partial u^i$) and the curvature tensor is

$$R_{i\bar{j}k\bar{l}} = G_{i\bar{s}}\partial_k \Gamma^{\bar{s}}{}_{\bar{j}\bar{l}}\,. \tag{3.6}$$

A displacement in $\mathcal{M}$ deforms the complex structure of Σ, so we expect the holomorphic differentials ζ_l to mix with the antiholomorphic ones. It is easy to show that the covariant derivative of a $(1,0)$ form gives a pure $(0,1)$ form:

$$\nabla_i \zeta_j = c_{ij}{}^{\bar{k}} \bar{\zeta}_{\bar{k}}\ ,\ \ c_{ij}{}^{\bar{l}} := i G^{k\bar{l}} \int \nabla_i \zeta_j \wedge \zeta_k\,, \tag{3.7}$$

and the relation with the curvature is

$$R_{i\bar{l}j\bar{k}} = -i\, c_{ijm} c^m{}_{\bar{k}\bar{l}}\,. \tag{3.8}$$

3.1 Number of supersymmetric solutions

We want to count the vacua that preserve $N = 1$ supersymmetry. In the limit $M_{Pl} \to \infty$, supersymmetric solutions are given by $\partial_i W_{\text{eff}} = 0$, where $\partial_i := \partial/\partial u_i$. As explained before, this limit corresponds to taking into account only a neighborhood of the singularity, so that supergravity effects are negligible.

Solutions to these equations may be viewed in two equivalent ways. If we want to stabilize at a particular point in the moduli space, $\partial_i W_{\text{eff}} = 0$ is an on-shell condition that restricts the possible values of the fluxes to a subspace. Indeed, since $\partial_i R$ gives by construction a basis of $H^{1,0}(\Sigma')$,

$$\partial_i W_{\text{eff}} = \int T \wedge \partial_i R = 0\ \ ,\ \ i = 1, \ldots, n$$

implies that

$$T\ =\ (N^R - \tau N^{NS})\,\tau_{P,\tilde{P}} + \sum_{i=1}^{g} (N_i^R - \tau N_i^{NS})\,\zeta_i\ =\ \sum_{i=1}^{n} (N_i^R - \tau N_i^{NS})\,\zeta_i\,. \tag{3.9}$$

On the other hand, a holomorphic differential is uniquely specified by giving its A-periods. Indeed, the B-periods are then functions of the period matrix:

$$\int_{B_j} T = \sum_{i=1}^{n} (N_i^R - \tau N_i^{NS}) \int_{B_j} \zeta_i\,. \tag{3.10}$$

The other possible point of view is that we can turn on arbitrary fluxes through all the cycles; this will lift almost all the degeneracy of the $N = 2$ supersymmetric moduli space, leaving only some number of $N = 1$ supersymmetric vacua. Therefore, if we specify arbitrarily both the A and B fluxes, (3.10) stabilizes the complex moduli of the curve:

$$\beta_j^R - \tau \beta_j^{NS} = -\sum_{i=1}^{n} (N_i^R - \tau N_i^{NS}) \int_{B_i} \zeta_j\,. \tag{3.11}$$

The ingredient that makes the number of vacua finite in compact Calabi-Yau manifolds is the tadpole cancellation condition [4]. There is no such constraint in the noncompact case, since the flux can go off to infinity. However, the fluxes cannot be arbitrarily large, because once their associated energy is of order M_{Pl}, the noncompact approximation breaks down: our local variety will be mixed with far away cycles in the CY. Therefore, in counting the total number of vacua, we have to impose by hand a tadpole condition. By analogy with the compact case [14], we require that

$$\frac{i}{2\mathrm{Im}\tau}\int_{\Sigma'} T\wedge\bar{T} = L\,. \tag{3.12}$$

Using the on-shell formula (3.9) and recalling (3.3), the tadpole condition becomes

$$0 \leq L = \frac{1}{2\mathrm{Im}\tau}\, G_{i\bar{l}} U^i \bar{U}^l \leq L_* \tag{3.13}$$

where $U^i := N_i^R - \tau N_i^{NS}$. L_* is the maximum value of L, fixed by data of the compact CY that we choose to embed (2.2).

From (3.13), the counting of supersymmetric vacua may be rephrased in terms of the geometry of Σ: over each point (u^k) in moduli space we have a 'solid sphere' $U^i(u)$, with volume L_*. Each of these allowed points determines a point in flux space; the number of such points will give the number of supersymmetric vacua. Furthermore, (3.13) shows why degeneration limits may produce an infinite number of vacua: if $G_{i\bar{l}}$ develops a null direction, the tadpole condition will not bound the number of flux points. In other words, from this analysis it is not clear how configurations where one flux goes to infinity and another goes to minus infinity, in a correlated way such that $L \geq 0$ stays finite, will be ruled out. The gauge theory analysis will shed light on this point.

Finally, even with the tadpole condition, the number of solutions to the equations of motion (3.11) with continuous fluxes β_i will be infinite. Fortunately, there is a simple way out of this problem. Recall that the noncompact hyperelliptic curve should be considered as part of a compact CY. Instead of parametrizing the fluxes with arbitrary energy scales Λ_i, we take them to be integers. Then (2.16) will *fix* the energy scales at particular values, depending on the fluxes. This approach was also taken in [14] to study the consequences of the Klebanov-Strassler solution [15] and leads to the usual exponentially large hierarchies of energy scales, as we show later.

Now we have all the elements to count vacua on complex curves with punctures; the derivation of the formula for the density of vacua continues as in [4]: the number of supersymmetric vacua is given by

$$\begin{aligned} N_{\mathrm{vac}}(L\leq L_*) = &\int_0^\infty dL\,\theta(L_*-L)\sum_{N_R,N_{NS}}\delta\Big(L-\frac{1}{2\mathrm{Im}\tau}\,G_{i\bar{l}}U^i\bar{U}^l\Big)\times \\ &\times\int\Big(\prod_{i=1}^n d^2u^i\Big)\,\delta(\partial W) \end{aligned} \tag{3.14}$$

with

$$\delta(\partial W) := \prod_l \delta(\partial_l W)\,\delta(\partial_{\bar{l}} W^*)\,|\det\partial^2 W\,|\,.$$

Here,

$$\partial^2 W := \begin{pmatrix} \partial_l \partial_n W & \partial_l \partial_{\bar{n}} W^* \\ \partial_{\bar{l}} \partial_n W & \partial_{\bar{l}} \partial_{\bar{n}} W^* \end{pmatrix} . \tag{3.15}$$

Because of $\delta(\partial W)$, we can replace $\partial_l \to \nabla_l$ in (3.15).

The main simplification in the noncompact case is that, since $\nabla_l \bar{\zeta}_{\bar{n}} = 0$, $\nabla_l \partial_{\bar{n}} W^* = 0$, and then

$$|\det \partial^2 W| = \det \partial^2 W = |\det \nabla_l \partial_n W|^2 . \tag{3.16}$$

Therefore *the number of supersymmetric vacua coincides with the supersymmetry index*, which is topological and, as we shall see, much easier to compute. On the contrary, in the compact case, when gravity is not decoupled, the supersymmetric index gives just a lower bound to the number of vacua.

The final result is

$$N^C_{\text{vac}}(L_*) = \frac{(2\pi L_*)^{2n}}{\pi^n (2n)!} \int_{\mathcal{M}} \det(-R) \tag{3.17}$$

where $\det R := \det_{\bar{s}\bar{r}} \left(R^{\bar{s}}{}_{\bar{r}k\bar{l}} \, du^k \wedge d\bar{u}^l \right)$. As expected, this coincides with [4] when $M_{Pl} \to \infty$. The index C is introduced for clarity reasons, to mean that this is the result from the closed string side.

4. The dual gauge theory

In this section we construct the supersymmetric gauge theory which is dual to the previous gravity configuration. The analysis will be done along the lines of the Dijkgraaf-Vafa (DV) correspondence, based on geometric transitions connecting open and closed superstrings. However our situation is more general and will require additional techniques.

Let us first quickly review the DV case, which corresponds to the flux subspace $N_i^{NS} = 0$, $\beta_n^R = 0$ and $\beta_i^{NS} = \beta_n^{NS}$ for all $i = 1, \ldots, n-1$. The large N duality between open/closed topological strings was derived in [16]. The role of the holomorphic matrix model and the relation to $N = 1$ SYM was considered in [5, 8, 17]. On the other hand, in [18] the DV relation was derived purely from the field theory side, using the chiral ring relations and the Konishi anomaly.

Close to the semiclassical limit $|a_i^+ - a_i^-| \ll a_i$, $S_i \to 0$, the geometry (2.2) corresponds to a product of n independent deformed conifolds. They are cones over $S^3 \times S^2$ and, while the S^2s are collapsed to zero, the S^3s have finite size as measured by $S_i \neq 0$. In the geometric transition the n 3-spheres A_i are collapsed and we blow-up the conifolds at $x = a_i$ by introducing n $\mathbb{P}^1$'s. Then the RR fluxes N_i^R will disappear and, instead, we will have N_i^R D5 branes wrapping the corresponding $\mathbb{P}^1$s. The DV correspondence states that the large $N^R := \sum_{i=1}^n N_i^R$ limit of the closed string theory on the deformed threefold is equivalent to the open string theory on the resolved threefold, with the previous relation between RR fluxes and D5 branes.

$W(x)$ plays the role of a tree-level superpotential for the chiral superfield Φ in the $N = 2$ vector multiplet of a pure $U(N^R)$ SYM; this potential breaks $N = 2$ to $N = 1$. Classically, the number of vacua is given by the number of ways of choosing N_i^R eigenvalues

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of Φ equal to a_i, with $\sum_i N_i^R = N^R$. This breaks $U(N^R) \to \prod_i U(N_i^R)$. β_n^{NS} is the bare gauge coupling of $U(N^R)$, while c_i^R are relative changes in the θ-angles of the $U(N_i^R)$ factors [18]. Furthermore, the complex moduli measure gaugino condensation

$$S_i = -\frac{1}{32\pi^2}\langle \mathrm{Tr}\; W_\alpha W^\alpha P_i\rangle \tag{4.1}$$

(P_i projects onto $\Phi = a_i$).

4.1 Dualities and geometric transition

We return now to the general flux configuration (N_i^R, N_i^{NS}), $(\beta_i^R, \beta_i^{NS})$. Denote $N^R := \sum_{i=1}^n N_i^R$, $N^{NS} := \sum_{i=1}^n N_i^{NS}$ and $r = \gcd(N^R, N^{NS})$, i.e., $N^R = n_R r$ and $N^{NS} = n_{NS} r$ with n_R and n_{NS} relatively prime.

Consider first the effect of the geometric transition around the semiclassical regime. In the open string side we end with N_i^R D5-branes and N_i^{NS} NS5-branes wrapping the i-th $\mathbb{P}^1$. The β_i do not have a brane analogue since the B-cycles remain 3-cycles; their meaning will become clear later. Our aim is to find a gauge theory interpretation for these n (N_i^R, N_i^{NS}) 5-brane states. The basic requirement is that the infrared limit of this configuration shall be given by composite fields S_i with an effective superpotential

$$W_{\text{eff}} = \sum_{i=1}^n (N_i^R - \tau N_i^{NS})\frac{\partial \mathcal{F}}{\partial S_i} - 2\pi i \sum_{i=1}^n (\beta_i^R - \tau\beta_i^{NS}) S_i\,; \tag{4.2}$$

we omitted a $(-1/2\pi i)$ factor as compared to (2.11).

We expect each (N_i^R, N_i^{NS}) 5-brane to decay to r_i copies of an (n_i^R, n_i^{NS}) bound state [19]; here $N_i^R = n_i^R r_i$, $N_i^{NS} = n_i^{NS} r_i$ with n_i^R and n_i^{NS} coprime. However, the generic point in flux space will give n different types of bound states and it is hard to see how this may come from a unique UV gauge theory. Instead, the straightforward way of getting a gauge theory is if on each $\mathbb{P}^1$ we have *the same* type of bound state. Combining this with the requirement that the sum of fluxes ($N^R = n_R r, N^{NS} = n_{NS} r$) remains constant implies that we will have r copies of the bound state of type (n_R, n_{NS}) distributed over all the different $\mathbb{P}^1$s.

The physical mechanism that may be responsible for this is already known, namely, eigenvalue tunnelling in matrix models. Consider what happens when we tune the couplings a_k from (2.3) so that the n cuts come very close together: $y^2 = x^{2n} + \epsilon$, $\epsilon \to 0$. In this limit, the process of eigenvalue tunnelling between different cuts becomes relevant; this will result in RR flux transfer until we end with the same (n_R, n_{NS}) bound states in all the cuts. The tunnelling is explained by $D5$ branes wrapped around an S^3 interpolating between two S^2s in the resolved geometry [17]. This object is a domain wall from the EFT point of view, with tension $\partial\mathcal{F}/\partial S_i - \partial\mathcal{F}/\partial S_j$. After the tunnelling has taken place, we can tune back the couplings to their initial values.

We will now start to argue that the previous gauge theory is indeed the dual to our gravity configuration. The key elements entering into the argument are S-duality (decay to bound states) and moving the A_i cycles around, which is associated to an $\mathrm{Sp}(2n-2,\mathbb{Z})$ symmetry transformation. We work in the deformation side. Denote the deformed threefold defined in (2.1) and (2.2) by X_d; the limit $f_{n-1}(x) = 0$ is a singular CY X_s with (generically) conifold degenerations.

Recall that S-duality acts by $\mathrm{SL}(2,\mathbb{Z})$ transformations

$$\begin{pmatrix} F_3 \\ H_3 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F_3 \\ H_3 \end{pmatrix} , \; \tau \to \frac{a\tau + b}{c\tau + d} \, , \; ad - bc = 1 \, . \tag{4.3}$$

This doesn't change the geometry of the hyperelliptic curve (off-shell). On the other hand, the curve (2.2) has a symmetry group $\mathrm{Sp}(2n-2,\mathbb{Z})$ of matrices mixing the canonical cycles $(A_i\,,\,C_j)$. These transformations are generated by all the possible interchanges of the roots $a_i^\pm$. The generators are [20]

$$J = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} \; , \; \mathcal{A} = \begin{pmatrix} (A^t)^{-1} & 0 \\ 0 & A \end{pmatrix} \; , \; \mathcal{B} = \begin{pmatrix} \mathbb{I} & 0 \\ B & \mathbb{I} \end{pmatrix} . \tag{4.4}$$

$A \in GL(n-1,\mathbb{Z})$ and B is a symmetric matrix with integer coefficients. Note that $A_1 + \ldots + A_n = -C_P$ is invariant under $\mathrm{Sp}(2n-2,\mathbb{Z})$ because the loop around infinity doesn't change under monodromies of the roots.

The first step is to use S duality to set the total NS flux $N^{NS} = 0$ and hence $N^R = r$. The transformation doing this is

$$\begin{pmatrix} n_R r \\ n_{NS} r \end{pmatrix} \to \begin{pmatrix} a & -b \\ -n_{NS} & n_R \end{pmatrix} \begin{pmatrix} n_R r \\ n_{NS} r \end{pmatrix} = \begin{pmatrix} r \\ 0 \end{pmatrix} \tag{4.5}$$

for some integers (a,b) solving $an_R - bn_{NS} = 1$. We denote with tildes the transformed quantities after S duality.

Next we set $\tilde{N}_i^{NS} = 0$, $i = 1,\ldots,n-1$ with $\mathrm{Sp}(2n-2,\mathbb{Z})$ transformations. This is done with the 'diagonal' $\mathrm{SL}(2,\mathbb{Z})_i \subset \mathrm{Sp}(2n-2,\mathbb{Z})$ which mix the A_i and C_i cycles only:

$$\begin{pmatrix} \tilde{N}_i^{NS} \\ \tilde{c}_i^{NS} \end{pmatrix} \to \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \begin{pmatrix} \tilde{N}_i^{NS} \\ \tilde{c}_i^{NS} \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{c}_i'^{NS} \end{pmatrix} . \tag{4.6}$$

Primes refer to the transformed cycles. Symplectic transformations act in a complicated way on A_n; however, since we already fixed $N^{NS} = 0$ and $A_1 + \ldots + A_n$ is a symplectic invariant, we deduce that the combined application of (4.5) and (4.6) fixes *all* $\tilde{N}_i'^{NS} = 0$, $i = 1,\ldots,n$.

Summarizing, we have showed how $S \otimes \mathrm{Sp}(2n-2,\mathbb{Z})$ may be used to set all the NS fluxes through the A cycles to zero. The transformed axio-dilaton is $\tilde{\tau} = (a\tau - b)/(-n_{NS}\tau + n_R)$; the transformation of β_i will be analyzed shortly. Consider next the effect of the geometric transition [21] $X_d \to X_s \to X_r$ where X_r is the projective resolution blowing-up each conifold point in X_s to a $\mathbb{P}^1$; see figure 2. We end with r copies of *the same* 5-brane bound state (n_R, n_{NS}), wrapping the n $\mathbb{P}^1$ s. The gauge theory is then $\mathrm{U}(r) \to \prod_i \mathrm{U}(\tilde{N}_i'^R)$ where $\tilde{N}_i'^R$ is the number of (n_R, n_{NS}) bound states on the i-th $\mathbb{P}^1$. This is in agreement with our previous bound state reasoning in terms of eigenvalue tunnelling. The 3-cycles B_i don't collapse in the geometric transition, so in the open string side we still have the fluxes $(\beta_i^R, \beta_i^{NS})$.

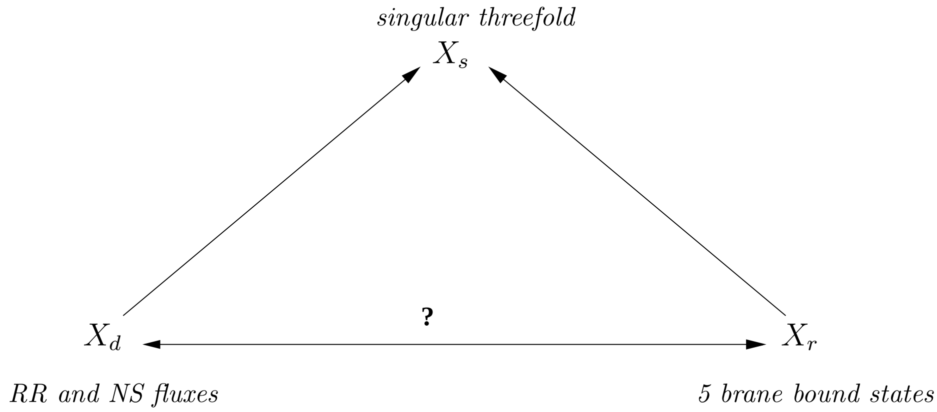

Figure 2: Geometric transition in the presence RR and NS fluxes.

4.2 Properties of the gauge theory

We don't know how to prove the duality $X_d \longleftrightarrow X_r$ conjectured in the previous subsection. Although the introduction of both RR and NS fluxes through the compact cycles is a natural extension of the Dijkgraaf-Vafa duality, an open topological string description of (n_R, n_{NS}) 5-brane bound states is not available. Instead, by computing the effective superpotential for both sides, we shall show that their predictions agree in the IR limit. As a further check, in section 6 we will prove that the gravity and gauge descriptions have the same number of degrees of freedom even in strongly coupled regimes, such as Argyres-Douglas singularities. (p, q) fivebranes wrapping an S^2 have also been considered in the different context of $N = 1^*$ SYM [22].

Consider how the effective flux superpotential (4.2) transforms under the $S \otimes \mathrm{Sp}(2n-2, \mathbb{Z})$ transformation given by (4.5) and (4.6):

$$\tilde{W}'_{\text{eff}} = \sum_{i=1}^{n} \tilde{N}_i'^R \frac{\partial \mathcal{F}}{\partial S'_i} - 2\pi i \sum_{i=1}^{n} \left(\frac{\beta_i'^R - \tau \beta_i'^{NS}}{n_R - \tau n_{NS}} \right) S'_i \,.$$

We made explicit the S duality transformation in the second term to exhibit the fractional dependence on $(n_R - \tau n_{NS})$; apart from this, $(\tilde{N}_i'^R,\ \beta_i'^R,\ \beta_i'^{NS})$ are all integers. Rename $\tilde{N}_i'^R \to N_i$ and drop all the primes:

$$W_{\text{eff}} = \sum_{i=1}^{n} N_i \frac{\partial \mathcal{F}}{\partial S_i} - 2\pi i \sum_{i=1}^{n} \left(\frac{\beta_i^R - \tau \beta_i^{NS}}{n_R - \tau n_{NS}} \right) S_i \,. \tag{4.7}$$

Here $(N_i,\ \beta_i^R,\ \beta_i^{NS})$ are arbitrary integers and shouldn't be confused with the original parameters appearing in (4.2).

Let us spell out the holomorphic properties of the gauge theory. Six dimensional gauge theories based on (p, q) 5-branes were studied for example in [23]. The situation here is more complicated, because the bound states are wrapping $\mathbb{P}^1$ s, and there is $(\beta_i^R, \beta_i^{NS})$ flux through such cycles.

Given that we have the same bound states (n_R, n_{NS}) in every $\mathbb{P}^1$, it is enough to study a single bound state wrapping a $\mathbb{P}^1$ and extending in four space-time dimensions. Since n_R

and n_{NS} are relatively prime, the S-duality transformation (4.5) maps the bound state to a single D5 brane. We denote with tildes the variables after the transformation. The DBI action is [19]

$$\begin{aligned} S &= S_{\text{kin}} + S_{CS} \\ S_{\text{kin}} &= -\mu_5 \int d^4x \int_{S^2} d\Omega_2\, \mathrm{e}^{-\tilde{\Phi}} \big[- \det(\tilde{G} + \tilde{B} + F) \big]^{1/2} \\ S_{CS} &= i\mu_5 \int \big[\tilde{C}_6 + (\tilde{B} + F) \wedge \tilde{C}_4 + \frac{1}{2}(\tilde{B} + F)^2 \wedge \tilde{C}_2 + \frac{1}{6}(\tilde{B} + F)^3\, \tilde{C}_0 \big] . \end{aligned} \tag{4.8}$$

$F := 2\pi\alpha' F_{ab}$ denotes the U(1) gauge field on the D-brane. Near the geometric transition point, where the S^2 shrinks, the holomorphic gauge coupling is given by

$$\tilde{\tau}_{YM} = (2\pi\alpha')^2 \mu_5 \bigg(\int_{S^2} \tilde{C}_2 - (\tilde{C}_0 + i\mathrm{e}^{-\tilde{\Phi}}) \int_{S^2} \tilde{B}_2 \bigg) . \tag{4.9}$$

The action for the (n_R, n_{NS}) bound state and the properties of its gauge theory follow from (4.8) and S-duality:

$$\begin{aligned} \tilde{\tau} &= \tilde{C}_0 + i\mathrm{e}^{-\tilde{\Phi}} = \frac{a\tau - b}{-n_{NS}\tau + n_R} , \\ \tilde{C}_2 &= aC_2 - bB_2 \ , \ \ \tilde{B}_2 = -n_{NS} C_2 + n_R B_2 \\ \tilde{G}_{ab} &= |n_R - n_{NS}\tau|\, G_{ab} \ , \ \ \tilde{C}_4 = C_4 \\ \tilde{B}_6 - \tilde{\tau}\tilde{C}_6 &= \frac{B_6 - \tau C_6}{n_R - \tau n_{NS}} . \end{aligned} \tag{4.10}$$

Noting that

$$\int_{S^2} (C_2 - \tau B_2) = \beta^R - \tau \beta^{NS} ,$$

the gauge coupling becomes

$$\tilde{\tau}_{YM} = \frac{\beta^R - \tau\beta^{NS}}{n_R - \tau n_{NS}} , \tag{4.11}$$

where we set $(2\pi\alpha')^2\mu_5 = 1$. This coincides exactly with the fractional holomorphic coupling derived from the flux side, eq. (4.7). Furthermore, once we map the system of (p, q) 5-branes to D5 branes, the arguments of [18] may be applied to this N=1 SYM theory to deduce that the effective superpotential has precisely the form given in (4.7). Generalizing to the case of n $\mathbb{P}^1$s, the gauge theory is $\mathrm{U}(r) \to \prod_i \mathrm{U}(N_i)$, $\sum_i N_i = r$, and each $\mathrm{U}(N_i)$ has a holomorphic coupling

$$\tau_i := \frac{\beta_i^R - \tau\beta_i^{NS}}{n_R - n_{NS}\tau} . \tag{4.12}$$

From our previous construction, it is clear that we didn't fix all the symplectic symmetries. In particular, we can still perform monodromies $S_i \to \mathrm{e}^{2\pi i}\, S_i$ corresponding to $B_i \to B_i + A_i$. This implies that τ_i is defined only modulo N_i or, equivalently,

$$\beta_i^R = 0, \ldots, n_R N_i - 1 ; \ \ \beta_i^{NS} = 0, \ldots, n_{NS} N_i - 1 . \tag{4.13}$$

We thus see that the information in the original brane system is not lost after the S-duality $(N^R, N^{NS}) \to (r, 0)$, but rather it is encoded in the holomorphic gauge couplings of the new theory.

It is worth noting that the holomorphic couplings τ_i, besides being fractional, they are also independent since we can choose arbitrary integers β_i. Equivalently from (2.16), each $\mathrm{U}(N_i)$ factor has an independent physical scale Λ_i. This situation is natural from the DBI action, but it cannot arise as the IR limit of the usual $N = 2$ $\mathrm{U}(r)$ SYM broken to $N = 1$ by the tree level superpotential $W(\Phi)$. Let us exhibit a simple generalization that may account for independent τ_i s. Coming from string theory, we won't require this UV gauge theory to be renormalizable, so we look for a modified kinetic term

$$\mathcal{L}_{\mathrm{kin}} \sim \int d^2\theta \, \mathrm{Tr} \left(W^\alpha W_\alpha \, f(\Phi) \right). \tag{4.14}$$

If $W(\Phi) = 0$, the gauge group is not broken and $f(\Phi_{\mathrm{class}}) = \tau_{YM}$ should give a unique gauge coupling. On the other hand, when we turn on the superpotential, the basic property of $f(\Phi)$ is that it should be equal to τ_i on the subspace $\Phi = a_i$. The matrix function that does this is simply constructed from the idempotents of the classical chiral ring:

$$E_i(\Phi) = \frac{\prod_{j \neq i}(\Phi - a_j \mathbb{I})}{\prod_{j \neq i}(a_i - a_j)}, \tag{4.15}$$

which satisfy $E_i(a_j) = \delta_{ij}$. Then we may define

$$f(\Phi) := \sum_{i=1}^{n} \tau_i E_i(\Phi). \tag{4.16}$$

The nonrenormalizable gauge theory (4.14) with this choice of $f(\Phi)$ gives independent gauge couplings in the infrared.

Another striking property of this brane system is the appearance of noncommutative dipoles in the UV. This is due to the NS fluxes through the $\mathbb{P}^1$ s. Such dipole deformations of the gauge theory have been recently considered in [24] for geometric transitions based on D5 branes. It would be interesting to try to extend this analysis to the case of (n_R, n_{NS}) 5-branes, although the supergravity description might be much more involved.

To summarize, using $S \otimes \mathrm{Sp}(2n-2, \mathbb{Z})$ in this section we mapped a general flux configuration to a gauge theory, after the geometric transition. All the flux parameters have a natural gauge interpretation; in particular the fluxes $(\beta_i^R, \beta_i^{NS})$ through the 3-cycles, which don't collapse after the transition, don't contribute brane degrees of freedom. They combine in a nontrivial way to determine the holomorphic gauge couplings of the different gauge factors.

5. Finiteness of vacua in the dual gauge side

The purpose of constructing a dual gauge theory to count flux vacua is that in such field theories the number of vacua is always finite. The geometric transition preserves this number. In the present section we show from the gauge theory side that N_{vac} is indeed finite.

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5.1 Proof of the finiteness of $N_{\rm vac}$

We begin by showing that the number of supersymmetric gauge vacua, i.e., solutions to $\partial W_{\rm eff}/\partial S_i$ from equation (4.7), is finite. As discussed before, this is based on the tadpole constraint

$$L = \sum_{i=1}^{n} N_i \tilde{\beta}_i^{NS} \, . \tag{5.1}$$

Here $\tilde{\beta}_i^{NS} = (n_R \beta_i^{NS} - n_{NS} \beta_i^R)$; also recall that $N_i := \tilde{N}_i^{\prime R}$, $\beta_i^R := \beta_i^{\prime R}$, $\beta_i^{NS} := \beta_i^{\prime NS}$.

We have to sum over all choices of fluxes satisfying (5.1). Here we run into the main obstacle. The reason why this could in principle diverge is that there may be flux configurations such that two terms in L grow in a correlated way to plus and minus infinity respectively, but keeping L finite and positive. This would give an infinite number of allowed flux points (and hence supersymmetric vacua).

This is the point where having a gauge theory based on the geometry (2.2) proves useful. In the gauge theory, $W_{\rm eff}$ is holomorphic in the couplings a_k, so the number of solutions to the equations $\partial W_{\rm eff}/\partial f_i = 0$ is invariant under smooth changes of the parameters, being protected by holomorphy.[2] An equivalent statement is that the number of vacua coincides with the dimension of the chiral ring of the theory, and such a quantity is independent of the gauge couplings. This topological behavior was already encountered in the gravity side, when we showed (section 3.1) that the number of supersymmetric vacua coincides with the supersymmetric index.

We now argue, from a variation of the a_k, that each term in L is in fact positive even around singularities. The discriminant locus consists of generic conifold points and higher codimension AD singularities. The later cannot be neglected because they have a higher 'weight' in the counting of degrees of freedom, as measured by $\det(R)$. Both situations will be exemplified in section 6.

Consider a point in moduli space $\mathcal{M}$ corresponding to the semiclassical limit. This is just the origin $S_i \to 0$ of $\mathcal{M}$. In this case the geometry is a product of independent conifold-like configurations. The effective superpotential follows from (4.7) using monodromy arguments [8]:

$$W_{\rm eff} = \sum_{i=1}^{n} N_i S_i \left(\log\left(\frac{\Lambda_0^3}{S_i}\right) + 1 \right) - 2\pi i \sum_{i=1}^{n} \left(\frac{\beta_i^R - \tau \beta_i^{NS}}{n_R - \tau n_{NS}} \right) S_i \, . \tag{5.2}$$

Denoting $\theta_i/2\pi := \mathrm{Re}(\tau_i)$ and $1/g_i^2 := \mathrm{Im}(\tau_i)$, the supersymmetric vacua may be written as

$$S_i = \exp(-i\theta_i/N_i) \exp(-2\pi/g_i^2 N_i)\, \Lambda_0^3 = \exp(-i\theta_i/N_i)\, \Lambda_i^3 \, . \tag{5.3}$$

Then counting vacua in the neighborhood of the conifold limit implies summing over fluxes giving $0 \le |S_i| \le ({\Lambda_i}^f)^3$.[3] Clearly this requires $\mathrm{sign}(n_R \beta_i^{NS} - n_{NS} \beta_i^R) = \mathrm{sign}(N_i^R)$, to avoid vacua exponentially far away from the origin. We therefore see that the number of

[2] Since off-shell the f_i don't depend on a_k, it is more convenient to take derivatives w.r.t. f_i and not S_i.
[3] $({\Lambda_i}^f)^3$ is some final energy scale associated to $\mathrm{U}(N_i)$.

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vacua around the semiclassical point is finite because each term in L is *separately* positive. Without loss of generality, we can just take all the fluxes to be positive.

The holomorphic dependence of $W_{\rm eff}$ on a_k implies that this is true for the whole moduli space. Indeed, every point in moduli space can be connected to the semiclassical limit by such a variation of couplings. Of course, strongly coupled limits may have quite complicated superpotentials, but we are interested in the number of vacua, which is a topological invariant.

For concreteness, we show this for $n = 2$. The hyperelliptic curve is

$$y^2 = (x^2 + g_1 x + g_0)^2 + f_2 x + f_1\,. \tag{5.4}$$

We only need to worry about singularities in $\mathcal{M}$ since it is known that $N_{\rm vac}$ is finite around smooth points. There are two types; the codimension one singularities are conifolds, and correspond to the semiclassical regime where we showed the finiteness of $N_{\rm vac}$. There is also a codimension two A_2 singularity. It corresponds to the singular limit of y:

$$y^2 = (x^3 - \delta u x - \delta v)(x - 1)\ \ ;\ \ \delta u\,,\ \delta v \to 0\,. \tag{5.5}$$

Three roots coincide at $x = 0$ giving two vanishing intersecting cycles, while the last one is fixed at $x = 1$. Comparing to (5.4), we find the 'double scaling' limit

$$f_1 = \delta v - \left(\frac{1}{8} + \frac{\delta u}{2}\right)^2\ ,\ \ f_2 = -\frac{1}{8} + \frac{\delta u}{2} - \delta v\,, \tag{5.6}$$

and, for the couplings,

$$g_1 = -\frac{1}{2}\ \ ,\ \ g_0 = -\left(\frac{1}{8} + \frac{\delta u}{2}\right). \tag{5.7}$$

To connect this to the semiclassical point, vary the couplings g_i from their previous double-scaled values to $g_i \gg f_i$, while keeping the f_i fixed at (5.6). Clearly, at the new point in $\mathcal{M}$ the semiclassical approximation is valid. This process is depicted in figure 3.

Therefore we have shown that any point in $\mathcal{M}$ can be connected to the conifold limit by a smooth variation of the a_k. In other words, the gauge theory tells us how to do, on every point in moduli space, a change of variables $S_i(a_k) \to S_i(\tilde{a}_k)$ such that: (i) each term in L is explicitly positive and (ii) the number of supersymmetric vacua doesn't change. Furthermore, since we can work in a regime $f_i \to 0$ by tuning $a_i \gg f_i$, we can always do power-series expansions and hence the change of variables is continuous. This maps compact regions to compact regions, assuring that the number of vacua doesn't diverge.

The meaning of this transformation becomes transparent if we consider the chiral ring. It is generated by idempotents and nilpotents [25]. If we move around the moduli space S_i by changing the couplings until we encounter a singularity, the result on the chiral ring is that some idempotents become nilpotents. The total number of generators is conserved in the process.

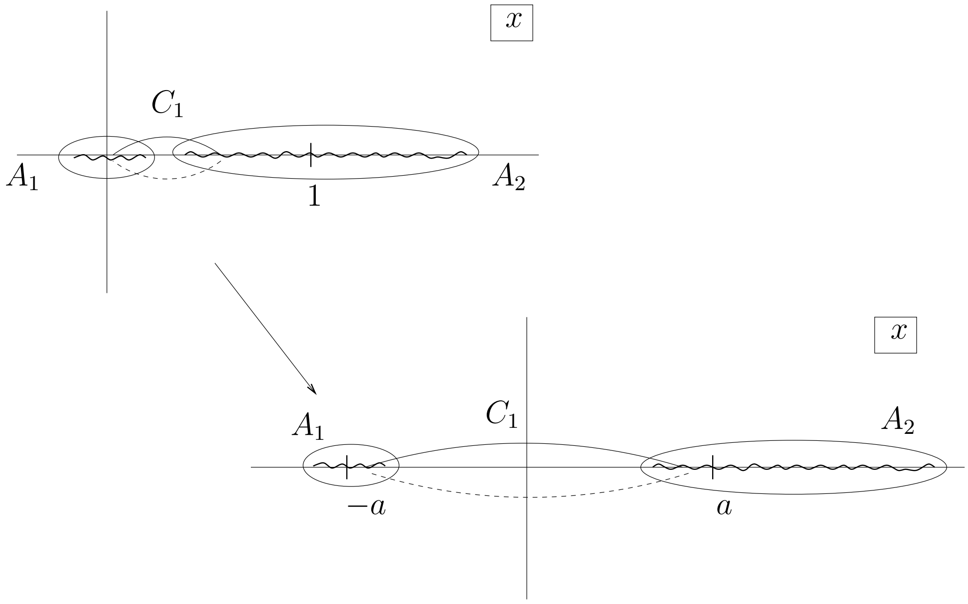

Figure 3: Holomorphic change of couplings that connects the AD point and the semiclassical limit.

5.2 Formula for $N_{\rm vac}(L_*)$

In order to compare with the gravity side result (3.17), we next compute the number of supersymmetric gauge vacua around an arbitrary point in $\mathcal{M}$. As argued before, holomorphy implies that we can as well compute it around the semiclassical limit.

Because of the monodromies leading to (4.13), at fixed N_i, the number of vacua is

$$N_{\rm vac}(\{N_i\}) = (n_R n_{NS})^n \prod_{i=1}^{n} N_i^2\,; \tag{5.8}$$

the N_i satisfy $\sum_i N_i = r$. This is quite different to the result from a standard $N=1$ SYM, $\prod_i N_i$. Eq. (3.17) includes an integration over a region in moduli space. We need to specify the analogous condition in the gauge side. It is associated to the RG flow of the gauge theory from the cutoff Λ_0 up to some IR energy scale Λ^f. For concreteness, we compute $N_{\rm vac}$ for the simplest case, namely when each $\mathrm{U}(N_i)$ flows up to a scale ${\Lambda_i}^f$. In other words, we assume that we are integrating on disks $0 \le |S_i| \le ({\Lambda_i}^f)^3$.

The renormalization of gauge couplings (2.16) applied to the case (4.7) gives

$$\frac{\tilde\beta_i^{NS}}{n_R^2 + n_{NS}^2} = \frac{1}{2\pi} N_i \log\left(\frac{\Lambda_0}{\Lambda_i}\right)^3 . \tag{5.9}$$

Here we set, for simplicity, $C_0 = 0$, $g_s = 1$. This is possible because in the noncompact model the axio-dilaton is fixed and behaves as a coupling; therefore $N_{\rm vac}$ cannot depend on it. Since we are summing the degrees of freedom with $0 \le \Lambda_i \le {\Lambda_i}^f$, (5.9) implies

$$\tilde\beta_i^{NS} \ge \frac{1}{2\pi}(n_R^2 + n_{NS}^2) N_i \log\left(\frac{\Lambda_0}{{\Lambda_i}^f}\right)^3 . \tag{5.10}$$

Replacing in the gauge tadpole condition (5.1),

$$(n_R^2 + n_{NS}^2) \sum_{i=1}^{n} N_i^2 \log\left(\frac{\Lambda_0}{{\Lambda_i}^f}\right)^3 \leq 2\pi L \,. \tag{5.11}$$

Once we fix arbitrary (N_i), the dual fluxes $(\tilde{\beta}_i^{NS})$ are integers satisfying the diophantine equation (5.1). This has solutions iff $\gcd(N_i)|L$; the number of integer solutions is of course infinite, but we argued that $\mathrm{sign}(N_i) = \mathrm{sign}(\tilde{\beta}_i^{NS})$. So we take the fluxes to be positive, and multiply the number of vacua by 2^n. The number of positive solutions to the tadpole constraint will be denoted by $b_+(\{N_i\})$. For large L, this number is typically of order 1.

Combining all the previous elements, the total number of supersymmetric vacua is

$$N_{\rm vac}(L_*; \Lambda^f) = 2^n \sum_{L=0}^{L_*} \sum_{n_R,\, n_{NS}\, coprime} (n_R n_{NS})^n \sum_{\{N_i\}:\, gcd(N_i)\,|L} \left[\prod_{i=1}^{n} N_i^2\right] \times$$
$$\times\, b_+(\{N_i\}) \,\cdot\, T(N_i; n_R,\, n_{NS})\,. \tag{5.12}$$

The notation here is the following. The sum on (n_R, n_{NS}) is over coprime integers. The sum on (N_i) should be done over inequivalent fluxes with respect to the residual symplectic transformations; indeed, some generators in (4.4) were not fixed by the mapping to the region $(N_i^R, N_i^{NS}) \to (N_i^R, 0)$. Also, recall that $b_+(\{N_i\})$ is the number of positive solutions to the diophantine equation (5.1); for large L_*, it will give subleading contributions so, to a good approximation, we may set $b_+ \sim 1$. Lastly, $T(N_i; n_R,\, n_{NS})$ specifies the region in flux space over which we are summing vacua. For instance, if we integrate on disks of radius $({\Lambda_i}^f)^3$, (5.11) gives the Heaviside function

$$T(N_i; n_R,\, n_{NS}) = \Theta\left(2\pi\, L - (n_R^2 + n_{NS}^2) \sum_{i=1}^{n} \log\left(\frac{\Lambda_0}{{\Lambda_i}^f}\right)^3 N_i^2\right). \tag{5.13}$$

6. Examples

In this section we compare the formulas (3.17) and (5.12) for $N_{\rm vac}$ in the gravity and gauge side, respectively. This is done for the conifold and Argyres-Douglas degenerations.

6.1 Example 1: the conifold

Gravity side. We start by considering the case of a single deformed conifold in the closed string side. The total number of vacua for the conifold has been computed in [26] in the context of F-theory compactifications. Here we quickly summarize the result for fixed axio-dilaton.

There is only one compact cycle (A), and a dual noncompact one (B). From monodromy arguments,

$$\int_A \Omega = z\ , \quad \int_B \Omega = \frac{1}{2\pi i}\, z \log\left(\frac{\mu}{z}\right) + \dots\,.$$

z is the complex modulus (here we don't use S to make clear the distinction between the gravity and gauge side) and μ is a constant added for dimensional reasons. It depends on the cutoff necessary to regulate the B-integral. Further, the dots refer to analytic terms in z.

Replacing in (3.4) and then in (3.6),

$$G_{z\bar{z}} \approx c \log\left(\mu^2/|z|^2\right)\ ,\ \ R^z{}_{zzz} = -\frac{1}{|z|^2\left(\log \mu^2/|z|^2\right)^2}\,.$$

For $z \to 0$, $G \ll R$ and hence $\det(-R-\omega) \approx \det(-R)$, in agreement with the deduced result (3.17). Integrating on $0 \le |z| \le R$, the number of supersymmetric vacua for fixed axio-dilation is

$$N^C_{\rm vac}(L_*) = \frac{2\pi^2\, L_*^2}{\log\frac{\mu}{R}}\,. \tag{6.1}$$

The superindex C reminds us that this is the result from the closed string side.

Gauge theory side. Next we calculate in detail the result from (5.12). From the gauge theory side, the conifold corresponds to the semiclassical limit of the superpotential with $n=1$: $W'(x) = x$ and from (2.15), $f_{n-1}(x) = f_1 = -4S$, where we are setting $g_n = 1$. There are N vacua satisfying

$$|S| = \mathrm{e}^{-2\pi/g^2N}\,\Lambda_0^3 := \Lambda^3$$

and we have to compute the number of vacua with $|S| \le \Lambda_f^3$ for some final energy scale Λ_f. From the running of the gauge coupling,

$$\tilde{\beta}^{NS} \ge \frac{1}{2\pi}\,(n_R^2 + n_{NS}^2)\,N\log\left(\frac{\Lambda_0}{\Lambda_f}\right)^3\,.$$

The meaning of this formula is that the gauge theory analogue of integrating a given modulus on a disk is the RG flow of the gauge coupling from the UV cutoff up to a final energy scale given by the radius of the disk.

The number of vacua for given L is then given by

$$N^O_{\rm vac}(L) = 2\sum_{N|L} N^2 \sum_{n_R,\,n_{NS}\;coprime} n_R n_{NS}\,\Theta\!\left(\frac{2\pi L}{\log\,(\Lambda_0/\Lambda_f)^3 N^2} - (n_R^2+n_{NS}^2)\right); \tag{6.2}$$

we multiply by 2 since we are considering only $N \ge 0$. The superindex O refers to the open string side.

The gravity result $\det(-R)$ arises in the continuum limit $L_* \gg 1$. Therefore we need to estimate the asymptotic behavior of $\sum_{L=0}^{L_*} N^O_{\rm vac}(L)$. We did this with a C++ program[4] that adds coprime numbers (modulo permutations) inside a disk of radius

$$\frac{2\pi L}{\log\,(\Lambda_0/\Lambda_f)^3 N^2}$$

and then sums over all the divisors of L, according to (6.2).

[4] We thank S. Lukic for help with this.

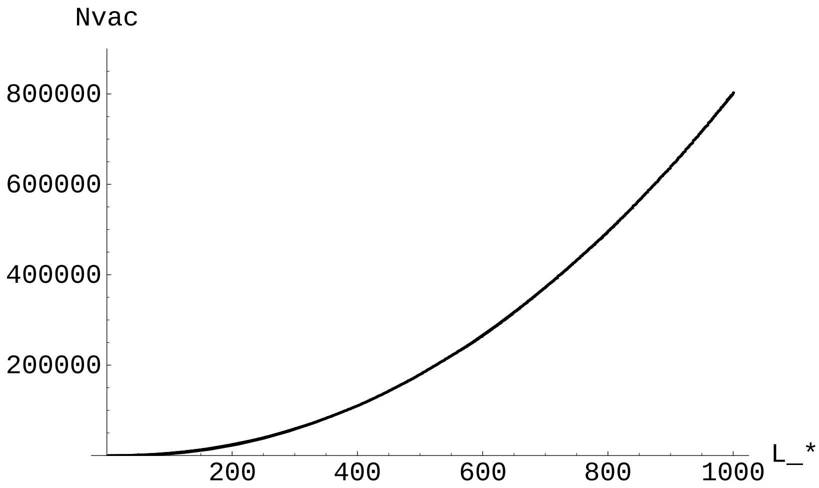

Figure 4: Plot of $N_{\text{vac}}(L_*)$ for the conifold, showing both the gravity and gauge side predictions, which agree almost exactly. We chose a scale $\log{(\Lambda_0/\Lambda_f)^3} = 8\pi$ to simplify the results.

Fitting the numerical predictions of $\log N_{\text{vac}}(L_*)$ for $L_* = 1000$, we deduce an asymptotic dependence $\log N_{\text{vac}}(L_*) \approx 2.017 \log{(L_*)}$. To leading order we find

$$N_{\text{vac}}(L_*) = \frac{8\pi}{\log{(\Lambda_0/\Lambda_f)^3}} \left(0.7852\, L_*^{2.017} - 12.370\, L_* \log L_*\right) . \tag{6.3}$$

The numerical results and the fit are shown in figure 4. We don't completely understand the subleading corrections to the gravity result. Even though we fit the numerical formula with $L_* \log L_*$, the power of L_* could be smaller.

Let us compare (6.1) and (6.3); we naturally identify $R := \Lambda_f^3$ and $\mu := \Lambda_0^3$ and both results match very well. The power 2.017 is a good approximation to the gravity result L_*^2. It turns out to be related to properties of the divisor functions $\sigma_k(n)$. The agreement is nontrivial, involving very different concepts in the gauge and gravity side. The crucial ingredients from the gauge side are the running of the gauge coupling and the correct tadpole condition. In other words, the gravity side with general fluxes has the same number of degrees of freedom as the SYM theory described in section 4.

6.2 Example 2: Argyres-Douglas singularities

Next we analyze some aspects of two-parameter models which arise from $n = 2$ superpotentials:

$$y^2 = (x^2 + g_1 x + g_0)^2 + f_2 x + f_1 . \tag{6.4}$$

The novel phenomenon for $n \geq 2$ is the appearance of Argyres-Douglas points, when three or more roots coincide; see (5.5). When intersecting cycles vanish simultaneously nonlocal dyons become massless. The physics is radically different to that of the conifold, giving rise to an interacting SCFT [13].

Unfortunately, the complications of the model forbid a straightforward analysis similar to the one done in the previous subsection. From the gravity side, the discriminant locus is a knot-like complex curve [13] with self-intersections; integrating over all the moduli space to get the total number of vacua is hence quite involved. On the other hand, in the gauge theory, the combinatorics present in formula (5.12) are equally complicated. Therefore we will only study the vicinity of the AD point and we will show that the number of vacua obtained from $\det R$ has the expected gauge theory scaling behavior.

Gravity side. [5]

The dynamics around the AD point is controlled by the small complex curve

$$w^2 = x^3 - \delta u x - \delta v \,. \tag{6.5}$$

The discriminant locus is

$$\Delta \;=\; 4(\delta u)^3 - 27(\delta v)^2 = 0\,, \tag{6.6}$$

which is not smooth; indeed

$$\Delta = 0 \;\;,\;\; \partial_{\delta u}\Delta = \partial_{\delta v}\Delta = 0$$

has solution ($\delta u = 0$, $\delta v = 0$). This is the Argyres-Douglas point [13]. Usual monodromy arguments used to construct the periods cannot be applied now, since the self-intersection is not normal. Therefore we need to blow-up (6.6). The general procedure is described in [27] and has been recently applied to our present situation in [28].

The normal-crossing variables close to the AD point turn out to be

$$\Delta := \frac{(\delta u)^3}{(\delta v)^2} - \frac{27}{4} \;\;,\;\; \eta := \frac{\delta v}{\delta u}\,. \tag{6.7}$$

The original discriminant locus corresponds to $\Delta = 0$; η is the scaling variable in the SCFT. By rescaling $x = \eta\tilde{x}$, $w = \eta^{3/2}\tilde{w}$ the dependence on η disappears; the dependence on Δ follows from the usual monodromy $\Delta \to \mathrm{e}^{2\pi i}\,\Delta$.

We do a symplectic transformation so that the small periods are (S_1, $\frac{\partial\mathcal{F}}{\partial S_1}$) and the large ones are (S_2, $\frac{\partial\mathcal{F}}{\partial S_2}$). The dependence on Δ and η is

$$S_1 \propto \eta^{5/2}\,\Delta\;,\;\; \frac{\partial\mathcal{F}}{\partial S_1} \propto \eta^{5/2}\,\Delta\log\Delta\,. \tag{6.8}$$

The large periods are analytic in Δ and η.

Replacing these expressions in (3.3) and (3.6), the density of vacua (3.17) around the AD point is

$$dN_{\rm vac} \;\propto\; \frac{L_*^4\, d^2\,\Delta\, d^2\eta}{|\eta|\,|\Delta|^{\,2}\,(\log|\Delta|)^3}\,. \tag{6.9}$$

We see that the density of vacua is integrable on a disk around $(\Delta,\eta) = (0,0)$. In particular, integrating on $0 \le |\Delta| \le \Lambda_f^3$ gives a total number of vacua

$$N^C_{\rm vac}(L_*,\Lambda_f) = \frac{2\pi^2 k\, L_*^4}{(\log{(\Lambda_0/\Lambda_f)^3})^2}\,. \tag{6.10}$$

[5]Done in collaboration with F. Denef and B. Florea.

This proves that the number of vacua around the AD singularity is finite. The constant k depends on analytic data from the long cycles; in general it cannot be computed using monodromy arguments.

The result (6.9) is of the general form encountered in the analysis of different singularities in [28]

$$dN_{\rm vac} \sim \frac{dz\, d\bar{z}}{|z|^2(\log|z|)^p} \tag{6.11}$$

where $z = 0$ denotes de discriminant locus (in normal crossing variables). We will now justify this behavior from the field theory point of view.

Gauge side. This example is quite interesting, since we have to use the map connecting the strongly coupled AD point to the semiclassical regime.

The procedure was described in section 5. We vary g_k from (5.7) to $W'(x) = x^2 - a^2$, while keeping f_i fixed at (5.6). The condition that we end in the semiclassical regime is $a \gg f_i$. Furthermore, we can set $\eta = 1$ by choosing a perturbation with $\delta u = \delta v$. Indeed, we only want to reproduce the divergence $1/(\log|\Delta|)^3$ associated to the 'physical' discriminant component $\Delta = 0$. Expanding for a large, the expression for S_i in terms of Δ is

$$S_1 \approx S_2 = \frac{i\pi}{4} - i\pi\left(\frac{27}{4} + \Delta\right). \tag{6.12}$$

Also, $S_1 - S_2 \sim \mathcal{O}(a^{-3/2})$. Therefore, to leading order in a, $S_1 = S_2$ and they depend linearly on Δ; up to a shift by a constant, the integral $0 \le |\Delta| \le \Lambda_f^3$ is hence translated to $0 \le |S_i| \le \Lambda_f^3$.

In this case, the gauge vacua formula (5.12) reads

$$\begin{aligned} N^O_{\rm vac}(L_*) = \sum_{L=0}^{L_*} \sum_{(N_1,\, N_2):\, gcd(N_i)|L} N_1^2\, N_2^2 \sum_{(n_R,\, n_{NS})\, coprime} \times \\ \times\, (n_R\, n_{NS})^2\, \Theta\left(\frac{2\pi L}{\log\,(\Lambda_0/\Lambda_f)^3} - (n_R^2 + n_{NS}^2)(N_1^2 + N_2^2)\right). \end{aligned} \tag{6.13}$$

A numerical evaluation shows that (6.13) has the same dependence as (6.10):

$$N^O_{\rm vac}(L_*) \approx \frac{2\pi^2}{(\log\,(\Lambda_0/\Lambda_f)^3)^2}\; 0.0235\, L_*^{4.060}\,, \tag{6.14}$$

for $L_* \approx 1000$. Subleading corrections should be taken into account, but their general dependence is hard to estimate.

This is a nontrivial check for the argument that we can map any complicated singularity to the conifold regime and equivalently count vacua there. Moreover, $n = 2$ is the smallest genus for which the symplectic transformations $\mathrm{Sp}(2n-2, \mathbb{Z})$ come into play to count gauge vacua.

7. Conclusions

In this paper we have shown that the number of supersymmetric vacua N_{vac} around ADE singularities of Calabi-Yau's in type IIB flux compactifications is finite. The argument is based on the existence of dual gauge theories, where finiteness may be shown.

Such singularities can be embedded in the noncompact CY (2.1) and it is crucial that some of the fields become nondynamical (couplings). The moduli are stabilized by turning on both RR and NS flux through the compact cycles. We then perform a geometric transition to connect this to the open string side.

The gauge theory is based on 5-brane bound states wrapping the resolved 2-cycles. Its main properties are obtained by applying S-duality to the DBI action on the resolved background. In particular, the theory has fractional gauge couplings τ_i; the couplings are independent and hence cannot come from a UV theory which is the usual $N = 1$ SYM with superpotential $W(\Phi)$.

More importantly, the effective superpotential of the field theory depends holomorphically on the couplings (a_k). The dimension of the chiral ring (N_{vac}) is thus invariant under changes δa_k. We used this property to map a generic point in field space (S_i) to the conifold limit, while preserving the number of vacua. In this semiclassical limit we showed that N_{vac} is finite.

Finally, we computed explicitly this number for the two simplest singularities, namely the conifold point and the $n = 2$ Argyres-Douglas point. The results from the gravity side and gauge side match. This agreement is nontrivial since it involves quite different concepts on both sides.

Let us compare both formulas. The gravity formula $\int \det(-R - \omega)$ relates supersymmetric vacua to the geometry of the moduli space. A simple topological interpretation [4] is that it gives the number of zeroes of the section $D_i W_{\text{eff}} \in \Gamma(\mathcal{M}\Omega \otimes \mathcal{L})$. Clearly, it is explicitly invariant under symplectic transformations. However, the analysis of singularities is not straightforward, in particular because the blow-up procedure becomes very involved as we analyze higher codimension singularities.

On the other hand, the physics underlying the gauge theory result is that of fractional instantons, bound states of 5-branes, RG flow of the gauge couplings and combinatorics between the matrix model cuts. The formula is explicitly finite after the mapping to the semiclassical region. As a result, we recognize the exponent p in (6.11) as the degree of the tree-level gauge superpotential $W(\Phi)$ in which the singularity may be minimally embedded. We should nevertheless point out that for $n \geq 3$ there remain symplectic generators that have to be fixed by further restricting the fluxes to a fundamental region and this is in general complicated. Another issue is that the combinatorics grows very rapidly with n and numerical computations become more difficult.

A technical point that could be addressed in the future is to understand better the origin of subleading corrections to the gravity formula. These appear because the flux space is in fact a lattice. The gauge theory approach might help in this direction.

We should note that the present results are based on the duality between the closed (deformed) and open (resolved) sides. We haven't been able to fully prove this, although

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we did show that both sectors have the same IR physics. It would be very interesting to continue this, perhaps with a supergravity analysis along the lines of [29]. If a lift to M-theory is possible, the geometric transition might reduce to a duality between M5 branes, as in the Dijkgraaf-Vafa context.

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